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Concave Programming in Control Theory

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Abstract. We show in the present paper that many open and challenging problems in control theory belong the the class of concave minimization programs. More precisely, these problems can be recast as the minimization of a concave objective function over convex LMI (Linear Matrix Inequality) constraints. As concave programming is the best studied class of problems in global optimization, several concave programs such as simplicial and conical partitioning algorithms can be used for the resolution. Moreover, these global techniques can be combined with a *local* Frank and Wolfe feasible direction algorithm and improved by the use of specialized stopping criteria, hence reducing the overall computational overhead. In this respect, the proposed hybrid optimization scheme can be considered as a new line of attack for solving hard control problems.

Computational experiments indicate the viability of our algorithms, and that in the worst case they require the solution of a few LMI programs. Power and efficiency of the algorithms are demonstrated for a realistic inverted-pendulum control problem.

Overall, this dedication reflects the key role that concavity and LMIs play in difficult control problems.

Key words: Fixed-order control, H_∞ synthesis, Robust control, Parametric uncertainty, Linear matrix inequalities, Global concave minimization, Frank and Wolfe algorithms

1. Introduction

A number of challenging problems in robust control theory fall within the class of rank minimization problems subject to LMI (convex) constraints. An important example is provided by the reduced-order H_∞ control problem. It has been shown in [10, 21, 31] that there exists a k-th order controller solving the H_∞ control problem of a plant with n-th order if and only if one can find a pair of symmetric matrices (X, Y) with dimension $n \times n$ such that for some H_∞ performance level γ the following holds.

$$(X, Y, \gamma)\mathcal{L}, \tag{1}$$

$$\text{Rank}\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leqslant n + k, \tag{2}$$

where $\mathcal{L}$ is a convex set defined by LMI constraints. More precisely, the constraint (1) has an expansion in the form

$$\mathcal{A}(x) := A_0 + \sum_{i=1}^{N} x_i A_i < 0,$$

where the x_i's are the decision variables linearly related to the original variables (X, Y, γ) and the A_i's are symmetric matrices. The inequality $\mathcal{A}(x) < 0$ must be understood in the semidefinite sense, that is, $\mathcal{A}(x)$ has only strictly negative eigenvalues.

The hardness of problem (1)–(2) stems from the rank condition (2) which is essentially nonconvex. Note that (2) is automatically satisfied with the case $k \geqslant n$ of arbitrary-order controllers and such the problem dramatically simplifies to (1), an LMI constraint that can be solved using highly reliable and efficient techniques in Semi-Definite Programming (SDP). Among such techniques are polynomial-time interior- point techniques extensively discussed in the monograph [28]. As it plays a central role in robust control theory, many researchers in the control community have devoted their efforts to developing heuristics and techniques for determining solutions to the class of nonconvex problems (1)–(2). See [12, 13, 15, 19] to cite a few and [14, 35, 36, 39] for methods that are related to global optimization techniques.

One of the main purposes of this paper is to show that not only problem (1)–(2) but also many other important and challenging problems in robust control theory can be recast as concave minimization problems. That is, prolems involving a concave functional subject to convex constraints consisting of LMIs. A sample list of such problems includes robust control and robust multi-objective problems based on any kind of scalings or multipliers, robust fixed- or reduced-order control problems, multi-objective Linear Parameter-Varying (LPV) control, reduction of LFT representations, and more generally any combination of such problems. These problems are generally difficult to deal with but exhibit some nice geometric concave structure that makes them more attractive and painless than general nonlinear optimization problems. Remarkably, though concave programming is the best studied class in global optimization since the pioneering paper [37], it seems to have escaped the control research attention, so that very little effort has been dedicated to the global approach to such problems. Another distinguished characteristic of the concave problems under study is that whenever feasible, optimality occurs only at zeros of the concave functional. In this respect, such problems can be reinterpreted as zero finding concave programs which significantly reduces the difficulty of the search. Thus, new stopping criteria which locate such zeros as fast as possible are of great interest and will be discussed briefly. Since local optimization algorithms are computationally much cheaper than global ones, it is also of interest to develop an adequate local optimization technique to determine a good enough initial value. The concave structure of the problem implies that the Frank and Wolfe algorithm should be very useful in that respect. As we shall see, it is guaranteed to generate strictly decreasing sequences for the objective functional and that the sequence of points is either infinite or reach a local optimal solution. Such local algorithms and stopping criteria are then combined with recently available concave programming methods [17, 18, 24, 38] to certify global optimality of the solutions or invalidate feasibility. The overall hybrid algorithm consists of a suitably built FW algorithm at the initializing stage associated with several alternative concave programming techniques in the central body.

The FW algorithm is much less costly but in return, is prone to non- global optimality. On the other hand, concave minimization techniques provide global optimal solutions but generally require intensive computations. Therefore, an important target of this paper is to maintain a reasonable computational cost by taking advantage of local and global techniques. Hence, the global concave programming techniques are used either to refine a local solution issued from the FW algorithm until global optimality is achieved or to provide a certificate of global optimality. We have paid special attention to the simplicial and conical Branch and Bound concave minimization methods [38] which respectively divide the feasible set into simplices and cones of decreasing sizes. The main thrust of these techniques is that they rely heavily on concavity and convexity geometric concepts which make them particularly appropriate for our problems. Each step of the proposed techniques exploits both the convexity of the constraint set and the concavity of the functional and also the fact that only zero optimal values are of interest. This allows large portions of the feasible set to be eliminated at each iteration. The most computationally demanding operation in each step comes down to solving one LMI program, hence the practicality of the methods. On the other hand, the stopping criteria mentioned above reveal very useful to further reduce the computational cost.

Intensive computational experiments indicate that the local solutions found by the FW algorithm are very close to optimality and are either certified global or quickly improved to optimality after a few iterations of the simplicial and conical techniques. The reader is referred to [4] and its extended version freely available upon request for other details and a catalog of examples.

The paper has a tutorial nature since it relies on existing results in the area of robust control with LMIs. See the bibliography section provided at the end. Our intention has been to point the optimization community attention to some other classes of problems and structures, which difficulties are encountered and what kind of techniques are likely to be used. For clarity and understanding of the concepts, we have repeated some of the proofs. The reader is referred to additional material when lengthy derivations are required. The remainder of the paper is organized as follows. Instrumental tools are introduced in Section 2. Control problems and their formulation as concave minimization programs are discussed in Section 3 to 5, from the simple stabilization problem up to the more sophisticated robust control problems. Section 6 briefly focuses on specific local and global techniques as well as stopping criteria for arriving at a complete resolution. Finally, a realistic control problem illustrates the formulations and techniques in Section 7.

The following definitions and notations are used throughout the paper. M^T is the transpose of the matrix M, and M^* denotes its complex- conjugate transpose. The notation TrM stands for the trace of M while $\mathcal{N}_M$ is any matrix, whose columns form a basis of the nullspace of M. For Hermitian or symmetric matrices, $M > N$ means that $M - N$ is positive definite and $M \geqslant N$ means that $M - N$ is positive semi- definite. The notation co$\{p_1, \ldots, p_L\}$ stands for the convex hull of the set $\{p_1, \ldots, p_L\}$. The notation vert(P) is used to denote the set of vertices of a polyhedron P. Simplices and cones are defined in the usual way. In symmetric block matrices or long matrix expressions, we use $\star$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$\star \begin{bmatrix} S & M \\ \star & Q \end{bmatrix} K \equiv K^T \begin{bmatrix} S & M \\ M^T & Q \end{bmatrix} K.$$

We shall also use $\nabla f(x)$ to denote the (row vector) gradient of the function f. Finally, in algorithm descriptions the notation X^k is used to designate the k-th iterate of the variable X. The notations int S and ∂S are used for the relative interior and the boundary of the set S.

2. Instrumental tools

As mentioned above, a number of challenging problems in robust control theory can be formulated as concave minimization programs. These reformulations are strongly based on the following lemmas which help simplifying the theoretical characterizations. The first one is the projection Lemma and allows the elimination of a matrix variable occurring linearly in some LMI expressions [10].

LEMMA 2.1 (Projection Lemma [10]). *Given a symmetric matrix $\Psi \in \mathbf{R}^{m\times m}$ and two matrices P, Q of column dimension m, the following problem*

$$\Psi + P^T X^T Q + Q^T X P < 0 \tag{3}$$

is feasible with respect to matrix X of compatible dimensions if and only if

$$\mathcal{N}_P^T \Psi \mathcal{N}_P < 0, \quad \mathcal{N}_Q^T \Psi \mathcal{N}_Q < 0, \tag{4}$$

where $\mathcal{N}_P$ and $\mathcal{N}_Q$ denote arbitrary bases of the nullspaces of P and Q, respectively.

REMARK 2.2. The LMI (3) admits a convex set of solutions. One can extract a particular solution using SDP techniques or more simply by direct matrix algebraic techniques. A detailed discussion is given in [6,10].

The following lemma is crucial for reducing the nonconvexity degree in the LMI approach to control problems.

LEMMA 2.3. *Given real symmetric matrices X and Y in $\mathbf{R}^{n\times n}$, there exist U in $\mathbf{R}^{k\times k}$ symmetric and M such that*

$$W^{-1} = \begin{bmatrix} U & M \\ M^T & X \end{bmatrix}^{-1} = \begin{bmatrix} * & * \\ * & Y \end{bmatrix}, \tag{5}$$

and

$$W > 0, \tag{6}$$

if and only if 0 *is the optimal value of the following problem of minimizing a concave function over a convex set*

$$\min Tr(X - Y^{-1} - VV^T) s.t. \tag{7}$$

$$V \in \mathbf{R}^{n\times k}, \begin{bmatrix} X & I & V \\ I & Y & 0 \\ V^T & 0 & I \end{bmatrix} \geqslant 0 \tag{8}$$

Proof. From the matrix completion result (see e.g. [18, 31]), (5) and (6) are equivalent

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geqslant 0, \tag{9}$$

$$\text{Rank}(X - Y^{-1}) \leqslant k. \tag{10}$$

But (10) holds true if and only if

$$X - Y^{-1} = VV^T, \quad V \in \mathbf{R}^{n\times k}.$$

for some matrix V of dimension $n \times k$. On the other hand, by a Schur complement argument, the convex LMI (8) gives $(X - Y^{-1} - VV^T) \geqslant 0$ which also implies $\text{Tr}(X - Y^{-1} - VV^T) \geqslant 0$. Then, we have $\text{Tr}(X - Y^{-1} - VV^T) = 0$ if and only if $X - Y^{-1} = VV^T$. Note that the objective (7) is concave in $X > 0, Y > 0$ and V. □

REMARK 2.4. Note that for $k = 0$, i.e. U disappears in (5) then problem (7), (8) is simplified to the minimization of the concave objective $\text{Tr}(X - Y^{-1})$ over the convex constraint (9). On the other hand, for $k = n$, the rank constraint (10) is automatically satisfied and thus (5), (6) are equivalent to the convex constraint (9).

The next lemma provides efficient means for assessing quadratic performance of a linear system and can be regarded as a generalization of a Lyapunov's stability theorem.

LEMMA 2.5 [32]. *The linear system*

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}w, x(0) = 0 \\ z &= \mathcal{C}x + \mathcal{D}w \end{aligned} \tag{11}$$

is internally stable (i.e. the matrix A is asymptotically stable) and the following quadratic performance condition

$$\int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} U & W \\ W^T & V \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} dt < 0, \quad \forall T > 0, \quad \forall w(t) \tag{12}$$

with given matrices W and symmetric $U > 0$ and V, holds if and only if there is a solution $\mathcal{P} > 0$ of the LMI

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} + \mathcal{C}^T W & \mathcal{C}^T \\ \mathcal{B}^T\mathcal{P} + W^T\mathcal{C} & V + W^T\mathcal{D} + \mathcal{D}^T W & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -U^{-1} \end{bmatrix} < 0 \tag{13}$$

Proof. Recall that the asymptotic stability of $\mathcal{A}$ means that the solution trajectories of $\dot{x} = \mathcal{A}x$ tend to zero as the time t tends to infinity, for arbitrary initial conditions. The celebrated Lyapunov theorem [26], which is a fundamental tool in stability theory states that $\mathcal{A}$ is asymptotically stable if and only if there is a solution $\mathcal{P} > 0$ of the LMI

$$\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} < 0 \tag{14}$$

Now, the implication (13)⇒(12) is easy to check. Indeed, $\mathcal{P} > 0$ in (13) particularly satisfies (14) which proves the asymptotic stability of $\mathcal{A}$.

Using a Schur complement, (13) is also equivalent to

$$\begin{aligned} &\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} + \mathcal{C}^T W \\ \mathcal{B}^T\mathcal{P} + W^T\mathcal{C} & V + W^T\mathcal{D} + \mathcal{D}^T W \end{bmatrix} + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} U[\mathcal{C} \quad \mathcal{D}] < 0 \\ &\Rightarrow \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \left\{ \begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} + \mathcal{C}^T W \\ \mathcal{B}^T\mathcal{P} + W^T\mathcal{C} & V + W^T\mathcal{D} + \mathcal{D}^T W \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} U \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < 0 \\ &\Leftrightarrow \frac{d}{dt} V(t) + \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} U & W \\ W^T & V \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} < 0 \end{aligned} \tag{15}$$

where $V(t) = x^T(t)\mathcal{P}x(t)$ and $(d/dt)V(t) = x^T(t)(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A})x(t) + 2x^T(t)\mathcal{P}\mathcal{B}w(t)$.

Noticing that $V(T) \geqslant 0, \forall T > 0$ and that with zero initial conditions $V(0) = 0$, and integrating (15) on the time interval $[0, T]$ yields (12).

The implication (12) ⇒ (13) is more delicate and follows from indefinite linear-quadratic control theory [34]. □

Probably, the most well-known variant of Lemma 2.5 is the so called Bounded Real Lemma characterizing the L_2-gain condition

$$\int_0^T [\gamma^{-1} z^T(t) z(t) - \gamma w^T(t) w(t)]\, dt < 0, \quad \forall T \geqslant 0, \tag{16}$$

for system (11) (see e.g. [1]), which means that the H_∞-norm of the transfer function $T(s) := \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ from w to z is bounded by γ. As shown in [10, 21, 31], this Lemma plays a crucial role in the LMI approach to the H_∞ control problem. Obviously, (16) is a particular case of (12) with $U = \gamma^{-1} I$, $W = 0$, and $V = -\gamma I$.

3. Stabilization problems

We begin our analysis of control problems that can be formulated as concave minimization problems by the static gain stabilization problem. This problem has a fairly simple formulation but retains the properties and difficulties of many problems examined throughout the paper.

3.1. STATIC STABILIZATION: OUTPUT FEEDBACK VS. STATE FEEDBACK

Consider a linear system which obeys the differential equation

$$\dot{x} = Ax + Bu,\ y = Cx, \quad A \in R^{n\times n} \tag{17}$$

where $x(t)$ is the state vector, $u(t)$ is the control signal and $y(t)$ is the measurement vector.

The static gain stabilization problem consists in the determination of a control signal

$$u = Ky, \tag{18}$$

where K is a static gain matrix such that the closed-loop system (17) and (18) is stable. That is, the state of the closed-loop system

$$\dot{x} = (A + BKC)x, \quad x(0) = x_0 \tag{19}$$

converges asymptotically to zero as time increases. This control objective admits an alternative matrix inequality characterization via Lyapunov Theorem [26]. As mentioned in the previous section, an equivalent formulation is therefore the existence of a symmetric matrix X such that the matrix inequalities

$$(A + BKC)^T X + X(A + BKC) < 0 \tag{20}$$

$$X > 0 \tag{21}$$

hold.

Condition (21) particularly implies that X is nonsingular and therefore $\mathcal{N}_{B^T X} = X^{-1}\mathcal{N}_{B^T}$. Then rewrite (20) in the form

$$(A^T X + XA) + C^T K^T B^T X + XBKC < 0, \tag{22}$$

and apply the projection Lemma 2.1 to (22). The problem is then easily reformulated as the matrix inequalities of (21) together with

$$\mathcal{N}_C^T (A^T X + XA)\mathcal{N}_C < 0, \tag{23}$$

$$\mathcal{N}_{B^T}^T X^{-1}(A^T X + XA)X^{-1}\mathcal{N}_{B^T} < 0$$
$$\Leftrightarrow \mathcal{N}_{B^T}^T (YA^T + AY)\mathcal{N}_{B^T} < 0 \tag{24}$$

with $Y = X^{-1}$. Thus, by Remark 2.4, this problem is equivalent to that 0 be the optimal value of the following concave program

$$\min \operatorname{Tr}(X - Y^{-1}) \text{ s.t. } (23), (24), (9). \tag{25}$$

Note that in the case of the state-feedback control (i.e. $C + I$), one just have to solve LMI (24) which is a convex SDP problem. This clarifies the hardness of output-feedback control problems in regard to their state-feedback versions.

REMARK 3.1 It is worth noticing that when a solution to (25) has been found, a solution K to the static gain stabilization problem is easily derived by solving (22), which for a given X becomes an LMI with respect to K. Here again, SDP solvers or direct algebraic techniques are useful for that purpose.

3.2. DYNAMIC CONTROL: FIXED-ORDER VS. FULL-ORDER

For reasons that are related to controllability and observability properties of the system triple (A, B, C), it may be that a static control is not sufficient for solving the stabilization problem just discussed. In such a case, we are led to using a dynamic controller $K(s)$ with prescribed order k (number of states), hence depending on the Laplace variable s, instead of a mere static gain K. In other terms, we have to find a dynamic controller $K(s)$ in the form

$$K(s) \begin{cases} \dot{x}_K = A_K x_K + B_K y \\ u = C_K x_K + D_K y \end{cases} \tag{26}$$

with $A_K \in \mathbf{R}^{k\times k}$, and transfer function $K(s) = C_K(sI - A_K)^{-1}B_K + D_K$.

With $x_a = [x^T \quad x_K^T]^T$, it is immediate to check that the closed-loop system (17) and (26) is nothing else than the following system

$$\dot{x}_a = A_a x_a + B_a u_a,\ y_a = C_a x_a, \tag{27}$$

$$u_a = K_a y_a, \tag{28}$$

with the notations

$$A_a = \begin{bmatrix} A & 0 \\ 0 & 0_k \end{bmatrix}, B_a = \begin{bmatrix} 0 & B \\ I_k & 0 \end{bmatrix}, C_a = \begin{bmatrix} 0 & I_k \\ C & 0 \end{bmatrix}, K_a = \begin{bmatrix} A_K & B_K \\ C_K & D_k \end{bmatrix}. \tag{29}$$

Our problem now becomes that of finding a static stabilizing control (28) with matrix gain K_a for system (27). We note that in (29) all matrices A_a, B_a, C_a are completely defined from A, B, C. The system (27) is called the augmented system for system (17).

Now, applying the result of Lemma 2.1 and in view of the trivial relations

$$\mathcal{N}_{B_a^T} = \begin{bmatrix} \mathcal{N}_{B^T} \\ 0 \end{bmatrix}, \mathcal{N}_{C_a} = \begin{bmatrix} \mathcal{N}_C \\ 0 \end{bmatrix},$$

we obtain the following characterization which is analogous to the conditions (23)–(24)

$$\begin{bmatrix} \mathcal{N}_C \\ 0 \end{bmatrix}^T (A_a^T X_a + X_a A_a) \begin{bmatrix} \mathcal{N}_C \\ 0 \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} \mathcal{N}_{B^T} \\ 0 \end{bmatrix} (Y_a A_a^T + A_a Y_a) \begin{bmatrix} \mathcal{N}_{B^T} \\ 0 \end{bmatrix} < 0 \tag{31}$$

$$Y_a = X_a^{-1} > 0. \tag{32}$$

The structure of the matrices in (29) implies that we can simplify (30)- -(31). Indeed, with the partition

$$X_a = \begin{bmatrix} X & N \\ N^T & E \end{bmatrix}, Y_a = \begin{bmatrix} Y & M \\ M^T & F \end{bmatrix}, X, Y \in \mathbf{R}^{n \times n}, E, F \in \mathbf{R}^{k \times k}, \tag{33}$$

we have

$$A_a^T X_a + X_a A_a = \begin{bmatrix} A^T X + XA & 0 \\ 0 & 0 \end{bmatrix}, Y_a A_a^T + A_a Y_a = \begin{bmatrix} AY + YA^T & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

and (30)–(31) becomes

$$\mathcal{N}_C^T (A^T + XA) \mathcal{N}_C < 0, \quad \mathcal{N}_{B^T}^T (AY + YA^T) \mathcal{N}_{B^T} < 0 \tag{35}$$

Finally, applying Lemma 2.3 to (32)–(33), we deduce the concave minimization formulation (with variables X, Y and V) of the fixed- order control problem as the concave programming problem

$$\min\{\mathrm{Tr}(X - Y^{-1} - VV^T) : \text{subject to LMIs (35) and (8)}\} \tag{36}$$

By similar arguments, various performance indexes can be handled such as H_2-norm performance, passivity constraints, and general quadratic constraints and their combinations. The reader is referred to [32] for a thorough discussion on these constraints.

Again, in the case of full-order control (i.e. $k = n$), by Remark 2.4, the dynamic stabilization problem reduces to the feasibility of LMIs (35) and (9), an easy convex problem. When solutions X, Y and V to problem (36) have been found, the explicit construction of the controller $K(s)$ can be performed using a standard procedure [10].

4. Fixed-order $\mathbf{H}_\infty$ synthesis problems

Stability is certainly a vital requirement in most control applications but it is generally not sufficient and additional practical specifications have to be taken into account. The H_∞ synthesis framework has received great attention in the last decade, mainly because it allows the formulation of a variety of practical specifications such as signal tracking, disturbance rejection, noise attenuation and loop-shaping constraints. A general formulation of the H_∞ synthesis control problem is as follows. We consider a linear time-invariant plant described in "standard form" by the state-space equations:

$$P(s) \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, & A \in \mathbf{R}^{n\times n} \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w, \end{cases} \tag{37}$$

where

- $u \in \mathbf{R}^{m_2}$ is the vector of control input(s)
- $w \in \mathbf{R}^{m_1}$ is a vector of exogenous inputs (reference signals, disturbance signals, sensor noise, etc.)
- $y \in \mathbf{R}^{p_2}$ is the vector of measurements
- $z \in \mathbf{R}^{p_1}$ is a vector of output signals related to the performance of the control system.

Let $T(s)$ denote the closed-loop transfer functions from w to z for some dynamic output-feedback control law $u = K(s)y$ defined by (26). Our goal is to compute a k-th order output-feedback controller (26) which meets the following design requirements

- *Internal stability:* for $w = 0$ all states of the closed-loop system (37) and (26) tend to zero as time tends to infinity.
- *performance:* the L_2-gain condition (16) is satisfied.

As for pure stabilization problems, it is possible to derive a matrix inequality characterization of this problem via a simple extension of Lyapunov theory [6]. The closed-loop system (37) and (26) can be rewritten in compact form as

$$T(s)\begin{cases}\dot{x}_{cl} = A_{c\ell}x_{cl} + B_{c\ell}w\\ z = C_{c\ell}x_{cl} + D_{c\ell}w\end{cases} \tag{38}$$

where

$$\left[\begin{array}{c|c} A_{c\ell} & B_{c\ell}\\ \hline C_{c\ell} & D_{c\ell}\end{array}\right] = \left[\begin{array}{cc|c} A + B_2D_KC_2 & B_2C_K & B_1 + B_2D_KD_{21}\\ B_KC_2 & A_K & B_KD_{21}\\ \hline C_1 + D_{12}D_KC_2 & D_{12}C_K & D_{11} + D_{12}D_KD_{21}\end{array}\right]. \tag{39}$$

Then, applying Lemma 2.5 to (38) with $U = \gamma^{-1}I$, $V = -\gamma I$ and $W = 0$, the above stability and performance requirements are met iff there exists a symmetric matrix $X_{c\ell}$ with

$$\begin{bmatrix} A_{c\ell}^TX_{c\ell} + X_{c\ell}A_{c\ell} & X_{c\ell}B_{c\ell} & C_{c\ell}^T\\ B_{c\ell}^TX_{c\ell} & -\gamma I & D_{c\ell}^T\\ C_{c\ell} & D_{c\ell} & -\gamma I\end{bmatrix} < 0, \quad X_{c\ell} > 0. \tag{40}$$

It is routine calculation to see that the first inequality in (40) can be rewritten in the form dictated by Lemma 2.1. Indeed, from expression (39) we can see that

$$\left[\begin{array}{c|c} A_{c\ell} & B_{c\ell}\\ \hline C_{c\ell} & D_{c\ell}\end{array}\right] = \left[\begin{array}{c|c} A_a + B_aK_aC_a & B_{1,a} + B_aK_aD_{21,a}\\ \hline C_{1,a} + D_{12,a}K_aC_a & D_{11} + D_{12,a}K_aD_{21,a}\end{array}\right] \tag{41}$$

where

$$K_a := \begin{bmatrix} A_K & B_K\\ C_K & D_K\end{bmatrix}, \quad A_a := \begin{bmatrix} A & 0\\ 0 & 0_k\end{bmatrix} \quad B_{1,a} := \begin{bmatrix} B_1\\ 0\end{bmatrix}, \quad C_{1,a} := \begin{bmatrix} C_1 & 0\end{bmatrix}$$
$$B_a := \begin{bmatrix} 0 & B_2\\ I_k & 0\end{bmatrix}, \quad C_a := \begin{bmatrix} 0 & I_k\\ C_2 & 0\end{bmatrix}, \quad D_{12,a} := \begin{bmatrix} 0 & D_{12}\end{bmatrix}, \quad D_{21,a} := \begin{bmatrix} 0\\ D_{21}\end{bmatrix}. \tag{42}$$

Then the first inequality in (40) can easily be rewritten as

$$\Psi + Q^TK_a^TP_{X_{c\ell}} + P_{X_{c\ell}}^TK_aQ < 0, \tag{43}$$

with the notations

$$P_{X_{c\ell}} := \begin{bmatrix} B_a^TX_{c\ell} & 0 & D_{12,a}^T\end{bmatrix}, \ Q := \begin{bmatrix} C_a & D_{21,a} & 0\end{bmatrix},$$

$$\Psi := \begin{bmatrix} A_a^TX_{c\ell} + X_{c\ell}A_a & X_{c\ell}B_{1,a} & C_{1,a}^T\\ B_{1,a}^TX_{c\ell} & -\gamma I & D_{11}^T\\ C_{1,a} & D_{11} & -\gamma I\end{bmatrix}.$$

Note that

$$P_{X_{c\ell}} = P \begin{bmatrix} X_{c\ell} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \text{ with } P = \begin{bmatrix} B_a^T & 0 & D_{12,a}^T \end{bmatrix},$$

from which we infer

$$\mathcal{N}_{P_{X_{c\ell}}} = \begin{bmatrix} X_{c\ell}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathcal{N}_P .$$

Hence, using Lemma 2.1, the existence of K_a in (43) is equivalent to the existence of $X_{c\ell} > 0$ such that

$$\mathcal{N}_Q^T \Psi \mathcal{N}_Q < 0 \Leftrightarrow \mathcal{N}_Q^T \begin{bmatrix} A_a^T X_{c\ell} + X_{c\ell} A_a & X_{c\ell} B_{1,a} & C_{1,a}^T \\ B_{1,a}^T X_{c\ell} & -\gamma I & D_{11}^T \\ C_{1,a} & D_{11} & -\gamma I \end{bmatrix} \mathcal{N}_Q < 0 \tag{44}$$

$$\mathcal{N}_{P_{X_{c\ell}}}^T \Psi \mathcal{N}_{P_{X_{c\ell}}} \Leftrightarrow \mathcal{N}_P^T \begin{bmatrix} X_{c\ell}^{-1} A_a^T + A_a X_{c\ell}^{-1} & B_{1,a} & X_{c\ell}^{-1} C_{1,a}^T \\ B_{1,a}^T & -\gamma I & D_{11}^T \\ C_{1,a} X_{c\ell}^{-1} & D_{11} & -\gamma I \end{bmatrix} \mathcal{N}_P < 0 \tag{45}$$

Similarly to Section 3.2, we use the following partition for $X_{c\ell}$ and $X_{c\ell}^{-1}$

$$X_{c\ell} = \begin{bmatrix} X & N \\ N^T & E \end{bmatrix}, X_{c\ell}^{-1} := \begin{bmatrix} Y & M \\ M^T & F \end{bmatrix} \tag{46}$$

Bases of the relevant subspaces are given as

$$\mathcal{N}_P = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \\ 0 & I \\ W_2 & 0 \end{bmatrix} \text{ with } \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathcal{N}_{[B_2^T \quad D_{12}^T]},$$

$$\mathcal{N}_Q = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \\ V_2 & 0 \\ 0 & I \end{bmatrix} \text{ with } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathcal{N}_{[C_2 \quad D_{21}]}.$$

This allows us to simplify (44)–(45) into

$$\star \left[\begin{array}{cc|c} XA + A^T X * & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_{[C_2 \quad D_{21}]} & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{47}$$

$$\star \left[\begin{array}{cc|c} YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_{[B_2^T \quad D_{12}^T]} & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{48}$$

Meanwhile, by virtue of Lemma 2.3, the existence of X and Y satisfying (46), (47) and (48) is equivalent to the zero-seeking concave program

$$\min \ \mathrm{Tr}(X - Y^{-1} - VV^T) : \ \text{LMIs (47) – (48), and (8).} \tag{49}$$

Again, by Remark 2.4, in the full-order case, $k = n$, the problem reduces to checking the feasibility of (9), (47)–(48), which is a standard (convex) SDP problem.

5. Robust control problems

A further requirement in control applications is that stability and performance are maintained in the presence of structured parametric uncertainties. This comes from the fact that plant's models are never perfectly known and one must account for uncertainties that invariably affect the state-space realization data. This is the problem investigated hereafter.

We are concerned with the robust control problem of an uncertain plant subject to LFT (Linear Fractional Transformation) uncertainty. In other words, the uncertain plant is described as

$$\begin{bmatrix} \dot{x} \\ z_\Delta \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w_\Delta \\ w \\ u \end{bmatrix} \tag{50}$$
$$w_\Delta = \Delta(t) z_\Delta,$$

where $\Delta(t)$ is an uncertain time-varying matrix-valued parameter and is usually assumed to have a block-diagonal structure in the form

$$\Delta(t) = \mathrm{diag}(\dots, \delta_i(t) I, \dots, \Delta_j(t), \dots) \in \mathbf{R}^{N \times N} \tag{51}$$

and normalized such that

$$\Delta(t)^T \Delta(t) \leqslant I, \quad t \geqslant 0. \tag{52}$$

Blocks denoted $\delta_i I$ and Δ_j are generally referred to as repeated-scalar and full blocks according to the μ analysis and synthesis literature [9, 8].

Clearly, the plant with inputs w and u and outputs z and y has state-space data entries which are fractional functions of the time- varying parameter $\Delta(t)$. This representation is fairly general and can encompass most practical situations. Here again, the meaning of u, w, z, y remains the same as that in Section 4.

For the uncertain plant (50)–(52) the robust control problem consists in seeking a linear time-invariant controller (26) such that for *all* parameter trajectories $\Delta(t)$

determined by (52), the closed- loop system (50)–(52) and (26) is internally stable and the L_2-gain conditions (16) is fulfilled.

As in Section 4, it is now well-known that such problems can be handled via a suitable application of Lemma 2.5. For a brief justification of this generalization, we need the following notation

$$\begin{aligned} \dot{x}_{c\ell} &= A_{c\ell} x_{c\ell} + B_{c\ell} \begin{bmatrix} w_\Delta \\ w \end{bmatrix} \\ \begin{bmatrix} z_\Delta \\ z \end{bmatrix} &= C_{c\ell} x_{c\ell} + D_{c\ell} \begin{bmatrix} w_\Delta \\ w \end{bmatrix}, \end{aligned} \tag{53}$$

where the state-space data $A_{c\ell}$, $B_{c\ell}$, $C_{c\ell}$ and $D_{c\ell}$ determine the closed-loop system (50) and (26) with the Δ loop $w_\Delta = \Delta(t) z_\Delta$ open.

We notice that checking condition (16) directly is generally intractable since all admissible Δ must be examined. Scalings or multipliers are therefore introduced to derive a relaxation of this problem, thus providing a sufficient condition. This relaxation requires the definitions of scaling sets compatible with the parameter structure given in (51). Denoting this structure as $\boldsymbol{\Delta}$, the following symmetric and skew-symmetric scaling sets can be introduced

$$\begin{aligned} S_{\boldsymbol{\Delta}} &:= \{S : S^T = S,\ S\Delta = \Delta S,\ \forall \Delta \text{ with structure } \boldsymbol{\Delta}\} \\ T_{\boldsymbol{\Delta}} &:= \{T : T^T = -T,\ T\Delta = \Delta^T T,\ \forall \Delta \text{ with structure } \boldsymbol{\Delta}\}. \end{aligned}$$

It is easily verified that with $S > 0$, the uncertain matrix Δ satisfies the quadratic constraints

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^T \begin{bmatrix} S & T \\ T^T & -S \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} \geqslant 0, \quad \forall \Delta \text{ s.t. } \Delta^T \Delta \leqslant I, \quad \text{with structure } \boldsymbol{\Delta}. \tag{54}$$

An equivalent form for (54) is also

$$\begin{bmatrix} z_\Delta \\ w_\Delta \end{bmatrix}^T \begin{bmatrix} S & T \\ T^T & -S \end{bmatrix} \begin{bmatrix} z_\Delta \\ w_\Delta \end{bmatrix} \geqslant 0, \quad \forall z_\Delta,\ w_\Delta = \Delta z_\Delta,\ \Delta^T \Delta \leqslant I, \\ \text{with structure } \boldsymbol{\Delta}. \tag{55}$$

With these definitions, a sufficient condition for (16) to hold for all possible Δ is the existence of $S > 0$ and T skew-symmetric such that

$$\int_0^T \left\{ \gamma^{-1} z^T(t) z(t) - \gamma w^T(t) w(t) + \begin{bmatrix} z_\Delta(t) \\ w_\Delta(t) \end{bmatrix}^T \begin{bmatrix} S & T \\ T^T & -S \end{bmatrix} \begin{bmatrix} z_\Delta(t) \\ w_\Delta(t) \end{bmatrix} \right\} dt < 0,$$

or alternatively,

$$\int_0^T \begin{bmatrix} z_\Delta(t) \\ z(t) \\ \hline w_\Delta(t) \\ w(t) \end{bmatrix}^T \left[\begin{array}{cc|cc} S & 0 & T & 0 \\ 0 & \gamma^{-1} I & 0 & 0 \\ \hline T^T & 0 & -S & 0 \\ 0 & 0 & 0 & -\gamma I \end{array}\right] \begin{bmatrix} z_\Delta(t) \\ z(t) \\ \hline w_\Delta(t) \\ w(t) \end{bmatrix} dt < 0. \tag{56}$$

Applying Lemma 2.5 to (53) with

$$U = \begin{bmatrix} S & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}, W = \begin{bmatrix} T^T & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} -S & 0 \\ 0 & -\gamma I \end{bmatrix}$$

The quadratic inequality (56) is equivalent to the existence of $X_{c\ell} > 0$, $S > 0$ and a skew-symmetric T such that

$$\begin{bmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & \star & \star \\ B_{c\ell}^T X_{c\ell} + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} C_{c\ell} & -\begin{bmatrix} S & 0 \\ 0 & \gamma I \end{bmatrix} + \left(\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} D_{c\ell} + \star \right) & \star \\ C_{c\ell} & D_{c\ell} & -\begin{bmatrix} S^{-1} & 0 \\ 0 & \gamma I \end{bmatrix} \end{bmatrix} < 0, \tag{57}$$

Exploiting this last condition which enforces both stability and performance for all admissible $\Delta(t)$, the next theorem gives a simplified characterization where the nonconvexity of the problem is clearly identified. Again the tools introduced in Section 2 are essential in the derivation. The reader is referred to references [2, 3, 16, 20, 29, 30, 33] for more details and additional results.

THEOREM 5.1 *Consider the LFT plant governed by* (50) *and* (52) *with* Δ *assuming a block-diagonal structure as in* (51). *Let* $\mathcal{N}_X$ *and* $\mathcal{N}_Y$ *denote any bases of the nullspaces of* $[C_2, D_{2\Delta}, D_{21}, 0]$ *and* $[B_2^T, D_{\Delta 2}^T, D_{12}^T, 0]$, *respectively. Then, there exists an* n*-th order controller such that condition* (57) *holds with* $X_{c\ell} > 0$ *and* $S > 0$ *and some* L_2*-gain performance* γ *if and only if there exist pairs of symmetric matrices* (X, Y), (S, Σ) *and a pair of skew-symmetric matrices* (T, Γ) *such that the structural constraints*

$$S, \Sigma \in S_{\mathbf{\Delta}} \text{ and } T, \Gamma \in T_{\mathbf{\Delta}} \tag{58}$$

hold and the matrix inequalities

$$\star \begin{bmatrix} A^T X + XA & XB_\Delta + C_\Delta^T T^T & XB_1 & C_\Delta^T S & C_1^T \\ B_\Delta^T X + TC_\Delta & -S + TD_{\Delta\Delta} + D_{\Delta\Delta}^T T^T & TD_{\Delta 1} & D_{\Delta\Delta}^T S & D_{1\Delta}^T \\ B_!^T X & D_{\Delta 1}^T T^T & -\gamma I & D_{\Delta 1}^T S & D_{11}^T \\ SC_\Delta & SD_{\Delta\Delta} & SD_{\Delta 1} & -S & 0 \\ C_1 & D_{1\Delta} & D_{11} & 0 & -\gamma I \end{bmatrix} \mathcal{N}_X < 0, \tag{59}$$

$$\star \begin{bmatrix} AY + YA^T & YC_\Delta^T + B_\Delta \Gamma^T & YC_1^T & B_\Delta \Sigma & B_1 \\ C_\Delta Y + \Gamma B_\Delta^T & -\Sigma + \Gamma D_{\Delta\Delta}^T + D_{\Delta\Delta} \Gamma^T & \Gamma D_{1\Delta}^T & D_{\Delta\Delta}\Sigma & D_{\Delta 1} \\ C_1 Y & D_{1\Delta} \Gamma^T & -\gamma I & D_{1\Delta}\Sigma & D_{11} \\ \Sigma B_\Delta^T & \Sigma D_{\Delta\Delta}^T & \Sigma D_{1\Delta}^T & -\Sigma & 0 \\ B_1^T & D_{\Delta 1}^T & D_{11}^T & 0 & -\gamma I \end{bmatrix} \mathcal{N}_Y < 0, \tag{60}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geqslant 0, \begin{bmatrix} S & 0 \\ 0 & \Sigma \end{bmatrix} > 0 \tag{61}$$

subject to the algebraic constraints

$$(S + T)^{-1} = (\Sigma + \Gamma), \tag{62}$$

or equivalently,

$$\begin{bmatrix} S & T \\ T^T & -S \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma & \Gamma^T \\ \Gamma & -\Sigma \end{bmatrix}, \tag{63}$$

are feasible.

Proof. See the Appendix A. □

Note that due to the algebraic constraints (62), the problem is nonconvex and has been even shown to have non-polynomial (NP) complexity. See [5] and references therein. Simpler instances of this problem as those considered in [27] are NP-hard. This feature is in stark contrast with the associated Linear Parameter-Varying control problem for which the LMI constraints (59)–(61) are the same but the nonlinear conditions (62) or alternatively (63) fully disappears. Also LMI (60) alone with $Y > 0$ is a characterization for the *full- information* control problem, a problem of independent interest, which is therefore convex.

The concave minimization formulation of (59)–(62) is following

LEMMA 5.2 *Introduce the concave LMI-constrained minimization program*

$$\mathbf{PB1}: \quad \min Tr(Z_1 - Z_3 Z_2^{-1} Z_3^T): \qquad (59)\text{–}(61), \tag{64}$$

$$\begin{bmatrix} Z_1 & Z_3 & S+T & I \\ Z_3^T & Z_2 & I & \Sigma+\Gamma \\ (S+T)^T & I & I & 0 \\ I & (\Sigma+\Gamma)^T & 0 & I \end{bmatrix} \geqslant 0. \tag{65}$$

Then, any feasible point to **Pb1** *which further satisfies*

$$Tr(Z_1 - Z_3 Z_2^{-1} Z_3^T) = 0, \tag{66}$$

is optimal and is a solution to the problem described in Theorem 5.1 and conversely.

Proof. This is a particular case of a result in [4]. □

Problem **PB.1** provides a characterization for full-order controllers $(k = n)$. If we further require that the controller be of reduced order $k < n$, then the problem should be formulated as

$$\min \ \mathrm{Tr}(Z_1 - Z_3 Z_2^{-1} Z_3^T) + \mathrm{Tr}(X - Y^{-1} - VV^T) : (59) - (61), (65), (8)$$

again as LMI-constrained zero-seeking concave problem.

Similar formulations can also be derived for the reduction of LFT representations and more generally to rank-constrained LMI problems and BMI (Bilinear Matrix Inequality) problems.

6. Solving methods

In this section we shall briefly describe some resolution algorithms for the concave programs **Pb1**. Other concave programs can be treated similarly. We see that **Pb1** is to check whether there exists

$$Z^* \in \chi = \{(Z_1, Z_2, Z_3) : \exists (X, Y, S, T_1 \Sigma, \Gamma) \text{ s.t. } (59) - (61), (65), \qquad (67)$$

satisfying $f(Z^*) = 0$ where $f(Z) := \mathrm{Tr}(Z_1 - Z_3 Z_2^{-1} Z_3^T)$ is concave. Such a Z^* when it exists will be called a *zero* of f. It is important to note that since f satisfies $f(Z) \geqslant 0, \forall Z \in \chi$, any zero of f is also a global optimal solution of

$$\min f(Z) : Z \in \chi, \qquad (68)$$

and consequently, our problem is much more computationally attractive than conventional concave programs in which minimal values of the cost function are unknown. In the methods presented hereafter, we can stop the search as soon as either such a zero is found in which case global optimality is ensured, or the minimum cost value is strictly positive in which case our problem has no solution. We refer the interested reader to [4] and its extended version for more details on the algorithms described in this section and extensive computational experiments. The extended version is available upon request to authors. Our intention here is the conception of an overall hybrid scheme, where the zero- seeking of f is accomplished by the combination of local optimization, global optimization and stopping tests.

6.1. A LOCAL FRANK AND WOLFE ALGORITHM AND STOPPING CRITERIA

For problem (68), the Frank and Wolfe (FW) algorithm at iteration k can be detailed as follows:
Find a steepest descent direction Z^{k+1} by solving the LMI program

$$\min Tr(G_1 Z_1 + G_2 Z_2 + G_3 Z_3) : \textit{(59)–(61), (65)}, \qquad (69)$$

where

$$G_1 := \frac{\partial f}{\partial Z_1} = I, \qquad G_2 := \frac{\partial f}{\partial Z_2} = Z_2^{k^{-1}} Z_3^{k^T} Z_3^k Z_2^{k^{-1}},$$
$$G_3 := \frac{\partial f}{\partial Z_3} = -2 Z_2^{k^{-1}} Z_3^{k^T},$$

If $f(Z^{k+1}) < f(Z^k)$ move to the next iteration. Otherwise stop the algorithm.

Since f is concave, the algorithm generates *strictly decreasing* sequences that can only terminate to a point satisfying the minimum principle local optimality conditions and the conventional line search at every iteration is bypassed. That is, a full step of one can be performed.

In order to further reduce the computational cost of the proposed techniques, we introduce some stopping procedures since in practical applications a perfect zero optimal value is never required which leaves some freedom to reduce the number of steps.

Given the current point of the algorithm determined by the variables (X^k, Y^k), (S^k, T^k), (Σ^k, Γ^k), Z_1^k, Z_2^k and Z_3^k our goal is to verify whether this point or a closely related point is a solution to the LMIs (59)–(61) subject to the algebraic constraint (62). In our new notation, our test takes the form

$$\text{LMIs (59)–(61)}, \tag{70}$$

$$(S^k + T^k)^{-1} = (\Sigma^k + \Gamma^k). \tag{71}$$

Note that in the course of the algorithm, the current point is not generally optimal so that the constraint (71) does not hold. It is, however, possible to terminate the program without reaching optimality. Our stopping criteria are based on the following perturbations techniques. We assume that a current feasible point of LMIs (59)–(61) and (65) is given. There exists a controller for which the conditions in Theorem 5.1 hold whenever one of the following perturbation techniques is successful.

- Compute $W = (S^k + T^k)^{-1}$ and update Σ^k and Γ^k using the substitutions

  $$\tilde{\Sigma}^k := \frac{W + W^T}{2}, \qquad \tilde{\Gamma}^k := \frac{W - W^T}{2}. \tag{72}$$

  Then, stop if new point passes the test (70).
- If previous test fails, then compute $W = (\Sigma^k + \Gamma^k)^{-1}$ and update S^k and T^k using the substitutions

  $$\tilde{S}^k := \frac{W + W^T}{2}, \qquad \tilde{T}^k := \frac{W - W^T}{2}. \tag{73}$$

  Then, stop if new point passes the test (70).
- or alternatively solve in P the perturbation problem

  $$(S^k + T^k + P)(\Sigma^k + \Gamma^k + P) = I \tag{74}$$

  or equivalently the generalized Riccati equation

  $$(S^k + T^k)P + P(\Sigma^k + \Gamma^k) + P^2 + (S^k + T^k)(\Sigma^k + \Gamma^k) - I = 0. \tag{75}$$

Then, stop if one of the solutions obtained with the substitutions

$$\tilde{S}^k = \frac{W_1 - W_1^T}{2}, \quad \tilde{T}^k = \frac{W_1 - W_1^T}{2}, \qquad (76)$$
$$\tilde{\Sigma}^k = \frac{W_2 + W_2^T}{2}, \quad \tilde{\Gamma}^k = \frac{W_2 - W_2^T}{2},$$

where

$$W_1 = S^k + T^k + P, \quad W_2 = \Sigma^k + \Gamma^k + P,$$

passes the test (70).

Note that without loss of generality, P in (75) can be selected as a general matrix with structure in conformity with the uncertainty structure $\mathbf{\Delta}$. The generalized Riccati equation (75) has a combinatoric of solutions that are easily computed using Hamiltonian techniques [7, 23, 25]. One can then easily extract a real smallest norm perturbation by combinatorial exploration. This task however will require extra computational efforts. It is thus recommended to use the first perturbation techniques alone for large size uncertainties.

6.2. GLOBAL SEARCH WITH THE SIMPLICIAL ALGORITHM

In view of the recent developments in global optimization, it seems that a BB method is the most suitable for our global search. The following analysis is useful to improve efficiency of the simplicial and conical BB methods.

Branching: The function f is not only concave in (Z_1, Z_2, Z_3) but is also linear in Z_1 with (Z_2, Z_3) held fixed, i.e. only (Z_2, Z_3) are the 'complicating' variables, responsible for the nonconvexity/hardness of the problem. The global search thus is concentrated on the reduced-dimensional space $\mathcal{Z}$ of variables (Z_2, Z_3). Accordingly, the feasible set can be interpreted as the projection of the convex set defined by the LMIs (59)–(61) and (65) on the space $\mathcal{Z}$. This space is partitioned into finitely many simplices. At each iteration, a partition simplex M is selected and subdivided further into several subsimplices according to the normal rule [38].

Bounding and terminating: Given a partition simplex M with vertices $u^1, u^2, \ldots, u^{N+1}$ (N is the dimension of $\mathcal{Z}$), the concavity of f and its linearity in Z_1 are further exploited in the search of a zero of f over $(Z_2, Z_3) \in M$. This is carried out through computing a lower bound $\beta(M)$ satisfying

$$\beta(M) \leqslant \nu(M) := \inf\{f(Z) : \quad Z \in \chi, (Z_2, Z_3) \in M\}. \qquad (77)$$

which is computed by

$$\min \operatorname{Tr}(Z_1)+\sum_{i=1}^{N+1} \lambda_i f(0, u^i) : \sum_{i=1}^{N+1} \lambda_i = 1, \lambda_i \geqslant 0, \left(Z_1, \sum_{i=1}^{N+1} \lambda_i u^i\right) \in \chi, \quad \min_{i=1,2,\dots,N+1} f(Z_1, u^i) \leqslant 0, \tag{78}$$

Of course, we can use the optimal solution $Z_1(M)$ and $\omega(M) = \sum_{i=1}^{N+1} \lambda_i(M)u^i$ of (78) not only for updating the best current value (upper bound) but also for the stopping test developed above to reduce the time of global search. Clearly, the partition sets M with $\beta(M) > 0$ cannot contain any zero of f and therefore are discarded from further consideration. On the other hand, the partition set with smallest $\beta(M) < 0$ can be considered the most promising one. To concentrate further investigation on this set, we subdivide it into more refined subsets. With a given tolerance $\varepsilon > 0$, the stop criterion of the BB algorithm is

$$\min_M \beta(M) \geqslant \varepsilon. \tag{79}$$

6.3. CONICAL ALGORITHM

Close scrutiny of the objective function properties $\operatorname{Tr}(Z_1 - Z_3 Z_2^{-1} Z_3^T)$ reveals the following.

- (i) If (Z_1, Z_2, Z_3) is the solution of **Pb1** with the zero optimal value then (tZ_1, tZ_2, tZ_3) with $t \geqslant 1$ is also a solution satisfying the same conditions. Thus, without loss of generality, we can set $\operatorname{Tr}(Z_1) = L$, with L a constant large enough.
- (ii) $Z_2 \geqslant I$ which means that we can use the change of variable $Z_2 \to Z_2 + \varepsilon I$ with $Z_2 \geqslant 0$ instead of $Z_2 > 0$.

As a consequence, problem **Pb1** can be reduced to minimizing the objective function

$$f(Z_2, Z_3) = L - \operatorname{Tr}(Z_3(Z_2 + \varepsilon I)^{-1} Z_3^T) \tag{80}$$

and LMIs (59)–(65) are changed accordingly using the substitution $Z_2 \to Z_2 + \varepsilon I$. The function f in (80) is concave in the cone $\mathcal{C}_+^{m_2} \times \mathcal{C}^{m_3}$ where $\mathcal{C}_+^{m_2}$ is the cone of nonnegative definite matrices with the same structure as Z_2 and $\mathcal{C}^{m_3}$ is the space of symmetric matrices having the same structure as Z_3. It is sufficient to take $\bar{\mathcal{Z}}$ as a large enough finite family of canonical cones approximating $\mathcal{C}_+^{m_2} \times \mathcal{C}^{m_3}$ with some tolerance. Perhaps, the most essential property of a concave function f is that its level sets $C_0 = \{Z = (Z_2, Z_3) \in \bar{\mathcal{Z}} : f(Z) \geqslant 0\}$ are convex and therefore an alternative formulation of our problem is to find $Z \in \chi \backslash \operatorname{int} C_0$ or else prove that $\chi \subset \operatorname{int} C_0$, where both χ, C_0 are convex sets. All these facts are taken into account in the global search with the conical algorithm based upon the so-called concavity cut or Tuy cut [37]. We omit the description here, the reader is referred [4] for details. By concentrating the search on the boundary of the feasible set, the conical algorithm better exploits the fact that the global minimum is attained at an extreme point and is therefore more efficient than the simplicial algorithm in the case of problem **Pb1**.

However, the simplicial algorithm is convenient for exploiting the partial linearity of the objective. For instance, in the case when all skew-symmetric matrices T and Γ vanish, the objective for **Pb1** can be reduced to the form

$$\operatorname{Tr}(S) - \operatorname{Tr}(\Sigma^{-1}), \tag{81}$$

which means that it is concave in Σ and linear in S. The simplicial algorithm can then be applied directly, with branching operations in the reduced Σ-space as previously. Thus in this case, the simplicial algorithm might be preferred.

7. Robust control of an inverted pendulum

This section provides an illustration of the local and global techniques introduced above. As mentioned in the introduction, the overall algorithm can be detailed as follows. The FW algorithm is computationally cheaper than simplicial and conical global techniques, and hence is used first to find a good suboptimal value γ. Then, the simplicial/conical algorithm are employed to further reduce γ, or to certify global optimality. The illustration consists of the robust control problem of an arm-driven inverted pendulum (ADIP) which is depicted in Figure 1. this is a two-link system comprising an actuated arm (first link) and a non-actuated pendulum (second link). The main control objective is to maintain the pendulum in the vertical position using the rotation of the arm. Moreover, this stabilization must be accomplished on a wide range of with respect to the angular position of the arm. A detailed description of the plant as well as the corresponding physical experiment is given in [22].

By selecting as state vector $x := [z \quad \dot{z} \quad r_x \quad \varphi_1]^T$, where r_x is the horizontal position of the arm tip (r_y is the vertical position), φ_1 and φ_2 are the angular positions of the arm and the pendulum, respectively, and $z := r_x + \frac{4}{3} l_2 \varphi_2$, The following simplified LFT state-space representation is obtained [22].

$$\dot{x} = Ax + B_\Delta w_\Delta + Bu, z_\Delta = C_\Delta x, w_\Delta = \Delta z_\Delta,$$

where the parameter structure is given as

$$\Delta := \begin{bmatrix} r_y & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & \varphi_2 \end{bmatrix}.$$

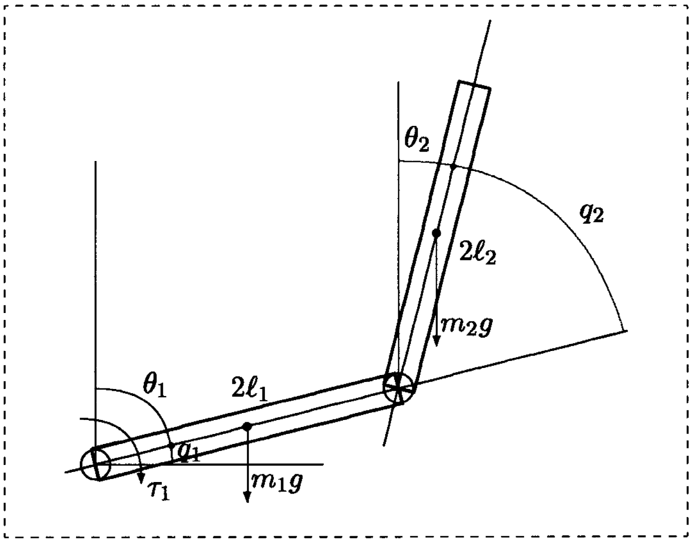

Figure 1. Inverted pendulum.

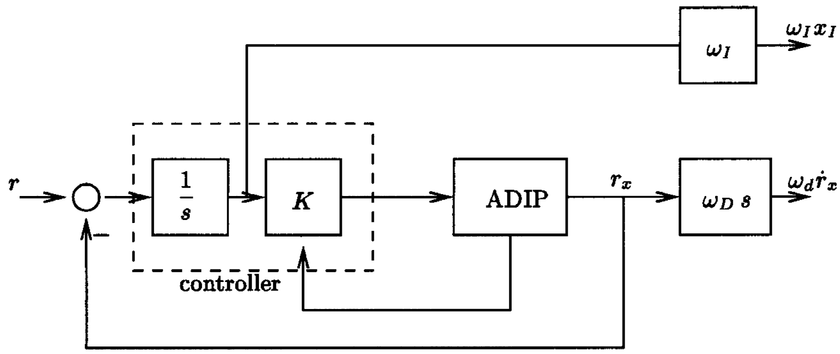

Figure 2. Synthesis structure for the inverted pendulum.

Therefore, the inverted pendulum admits LPV dynamics and can be controlled using either LPV or robust control techniques, as those considered in Section 5. Given an operating range for the inverted pendulum, the parameters are normalized such that $\Delta = \mathrm{diag}(\delta_1, \delta_2 I_2)$ with $|\delta_i| \leqslant 1, i = 1, 2$.

The synthesis structure used to achieve the design requirements is shown in Figure 2. It simply translates performance tracking $(\omega_I x_I)$ and high-frequency gain attenuation $(\omega_d \dot{r}_x)$. The numerical data of the synthesis interconnection are given in Appendix B.

Table 1 displays the performance of each algorithm in terms of number of iterations and cputime. The computations were performed on a PC with CPU Pentium II 330 Mhz and all LMI-related computations were performed using the *LMI Control Toolbox* [11]. Remember that the simplicial and conical algorithms are used only

Table 1. Performance of each algorithm

	FWA		SA		CA	
γ	# iter.	cputime	# iter.	cputime	# iter.	cputime
0.2	3	65.74 sec.	–	–	–	–
0.1910	10	148.03 sec.	–	–	–	–
0.1905	10	152.09 sec.	–	–	–	–
0.1904	2	56.08 sec.	–	–	–	–
0.1903	f	f	1	12.3 sec.	1	18.73 sec.
0.1838	–	–	2	84.80 sec.	1	18.95 sec.
0.18375	–	–	12(inf)	793.01 sec.	1	18.840 sec.
0.18370	–	–	1(inf)	13.03 sec.	1(inf)	16.04 sec.

FWA: Frank and Wolf algorithm; SA: simplicial algorithm; CA: conical algorithm;
f: the test fails; inf: no zero optimal value (infeas.)

after a the FW algorithm has failed ($\gamma = 0.1903$ in this case). The symbol ‘f’ indicates a failure of the FW algorithm to achieve the corresponding value of γ, first column, whereas the symbol ‘inf’ is used to specify infeasibility of γ.

From Table 1, we see that the performance found by the FW algorithm is within 5.5% of the global optimal value of γ. It is also worth noticing that with the same γ, there are many solutions obtained by the global algorithms. For instance, for $\gamma = 0.1838$, the scaling solutions with the simplicial and conical algorithms are given as

$$S = \begin{bmatrix} 1.2261 \times 10^{-5} & 0 & 0 \\ 0 & 0.5110 & -0.0231 \\ 0 & -0.0231 & 0.0042 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.0014 \\ 0 & 0.0014 & 0 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 1.2261 \times 10^{-5} & 0 & 0 \\ 0 & 0.1719 & 0.0010 \\ 0 & 0.0010 & 4.2145 \times 10^{-5} \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0073 \\ 0 & -0.0073 & 0 \end{bmatrix},$$

respectively. The optimal scalings with $\gamma = 0.18375$ and the conical algorithm are

$$\begin{bmatrix} 1.2264 \times 10^{-5} & 0 & 0 \\ 0 & 0.1748 & 0.0010 \\ 0 & 0.0010 & 3.3449 \times 10^{-5} \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0074 \\ 0 & -0.0074 & 0 \end{bmatrix}.$$

The optimal value of γ achieved with both the simplicial and conical algorithms are very close to that obtained using LPV synthesis where the parameter is not uncertain but known in real time ($\gamma = 0.1830$). This indicates that one will hardly find a better linear time-invariant controller for the specified control objectives. Similar realistic or randomized numerical experiments were conducted for other control problems, and indicate that the proposed techniques are very useful for solving such hard problems.

8. Concluding remarks

In this paper, we show that many important problems in robust control theory can be formulated as the minimization of a concave functional over a convex set determined by LMI constraints. The catalog given in this paper is by far non-exhaustive and many other control problems can be formulated in the same manner. In this respect, concavity appears to play a central role in a broad class of problems. This is the departure point which motivates the development of a comprehensive technique which provides a global solution of inherently difficult control problems. What is most promising is that NP-complexity never occurs in practical applications, so that the proposed algorithms are indeed useful and practical. These good results are obtained by exploiting a combination of a well-known method in classical differential optimization and more recent techniques in combinatorial concave minimization.

Appendix A

We note first that it is possible to simplify the proof by using the substitutions

$$\begin{aligned} &\mathcal{B}_1 := \begin{bmatrix} B_\Delta & B_1 \end{bmatrix}, \mathcal{C}_1 = \begin{bmatrix} C_\Delta \\ C_1 \end{bmatrix}, \\ &\mathcal{D}_{11} := \begin{bmatrix} D_{\Delta\Delta} & D_{\Delta 1} \\ D_{1\Delta} & D_{11} \end{bmatrix}, \mathcal{D}_{12} := \begin{bmatrix} D_{\Delta 2} \\ D_{12} \end{bmatrix}, \mathcal{D}_{21} := \begin{bmatrix} D_{2\Delta} & D_{21} \end{bmatrix} \end{aligned} \tag{82}$$

and

$$\mathcal{S} := \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}, \mathcal{T} := \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}. \tag{83}$$

where we assumed $\gamma = 1$ for further simplification.

As in Section 4, we see that the performance condition (57) is equivalent to

$$\Psi + P_{X_{c\ell}}^T K_a Q + Q^T K_a^T P_{X_{c\ell}} < 0, \tag{84}$$

where

$$\begin{aligned} \Psi &= \begin{bmatrix} A_a^T X_{c\ell} + X_{c\ell} A_a & X_{c\ell} B_{1,a} + C_{1,a}^T \mathcal{T}^T & C_{1,a}^T \\ B_{1,a}^T X_{c\ell} + \mathcal{T} C_{1,a} & -\mathcal{S} + \mathcal{T}\mathcal{D}_{11} + \mathcal{D}_{11}\mathcal{T}^T & \mathcal{D}_{11}^T \\ C_{1,a} & \mathcal{D}_{11} & -\mathcal{S}^{-1} \end{bmatrix} \\ P_{X_{c\ell}} &= \begin{bmatrix} B_a^T X_{c\ell} & D_{12,a}^T \mathcal{T}^T & D_{12,a}^T \end{bmatrix}, Q = \begin{bmatrix} C_a & D_{21,a} & 0 \end{bmatrix} \end{aligned}$$

with A_a, $B_{1,a}$, $C_{1,a}, \ldots$ defined in (42) and where $B_1, C_1, D_{11}, D_{12}, D_{21}$ are replaced with $\mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}$.

Thus by virtue of the Projection Lemma 2.1, (84) is equivalent to

$$W_{P_{X_{c\ell}}}^T \Psi W_{P_{X_{c\ell}}} < 0, \qquad W_Q^T \Psi W_Q < 0, \tag{85}$$

where $W_{P_{X_{c\ell}}}$, W_Q are any bases of the nullspaces of $P_{X_{c\ell}}$ and Q, respectively.

A basis of the nullspace of $P_{X_{c\ell}}$ is obtained as

$$\begin{bmatrix} X_{c\ell}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} W_P,$$

where W_P is any basis of the nullspace of $P = \begin{bmatrix} B_a^T & D_{12,a}^T \mathcal{T}^T & D_{12,a}^T \end{bmatrix}$.

An equivalent condition for the first inequality in (85) is thus $W_P^T \Phi W_P < 0$, with

$$\Phi = \begin{bmatrix} X_{c\ell}^{-1} A_a^T + A_a X_{c\ell}^{-1} & B_{1,a} + X_{c\ell}^{-1} C_{1,a}^T \mathcal{T}^T & X_{c\ell}^{-1} C_{1,a}^T \\ B_{1,a}^T + \mathcal{T} C_{1,a} X_{c\ell}^{-1} & -\mathcal{S} + \mathcal{T}\mathcal{D}_{11} + \mathcal{D}_{11}^T \mathcal{T}^T & \mathcal{D}_{11}^T \\ C_{1,a} X_{c\ell}^{-1} & \mathcal{D}_{11} & -\mathcal{S}^{-1} \end{bmatrix}. \tag{86}$$

From (42), it is easily inferred that bases of the null spaces of P and Q are obtained, respectively, as

$$W_P = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \\ 0 & I \\ W_2 & -\mathcal{T}^T \end{bmatrix}, \; W_Q = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \\ V_2 & 0 \\ 0 & I \end{bmatrix}, \tag{87}$$

where $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ and $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ are bases of the nullspaces of $\begin{bmatrix} \mathcal{B}_2^T & \mathcal{D}_{12}^T \end{bmatrix}$ and $\begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} \end{bmatrix}$, respectively. With the help of these notations, and exploiting the fact that the second row of W_Q is zero, the second projection in (85) with X defined from the partition (46) simplifies to (59) up to the congruent transformation

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{S} \end{bmatrix}.$$

Similarly, with T defined from the partition (46), the first projection in (85) reduces to

$$\begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & -\mathcal{T}^T \end{bmatrix}^T \begin{bmatrix} YA^T + AY & \mathcal{B}_1 + Y\mathcal{C}_1^T \mathcal{T}^T & Y\mathcal{C}_1^T \\ \mathcal{B}_1^T + \mathcal{T}\mathcal{C}_1 Y & -\mathcal{S} + \mathcal{T}\mathcal{D}_{11} + \mathcal{D}_{11}^T \mathcal{T}^T & \mathcal{D}_{11}^T \\ \mathcal{C}_1 Y & \mathcal{D}_{11} & -\mathcal{S}^{-1} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & -\mathcal{T}^T \end{bmatrix} < 0.$$

Computing this expression leads to

$$\begin{bmatrix} V_1^T(AY + YA^T)V_1 + V_2^T \mathcal{C}_1 Y V_1 + V_1^T Y \mathcal{C}_1^T V_2 - V_2^T \mathcal{S}^{-1} V_2 & * \\ \mathcal{B}_1^T V_1 + \mathcal{D}_{11}^T V_2 + \mathcal{T}\mathcal{S}^{-1} V_2 & -(S + \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^T) \end{bmatrix} < 0. \tag{88}$$

Finally, performing the changes of variable

$$\bar{\Sigma} = (\mathcal{S} + \mathcal{T}^T \mathcal{S}^{-1} \mathcal{T})^{-1}, \quad \bar{\Gamma} = -(\mathcal{S} + \mathcal{T} \mathcal{S}^{-1} \mathcal{T}^T)^{-1} \mathcal{T} \mathcal{S}^{-1},$$

with

$$\bar{\Sigma} := \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix}, \quad \bar{\Gamma} := \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently

$$(S + T)^{-1} = (\Sigma + \Gamma),$$

the congruent transformation

$$\begin{bmatrix} \bar{\Sigma} & 0 \\ 0 & I \end{bmatrix}$$

allows the identification of (88) with (60). To summarize, the problem is solvable if and only if (59)–(60), (61) and (62) have a solution such that $X_{c\ell}$ in (84) is positive definite. The latter condition is equivalent to the first LMI in (61) by Lemma 2.3. Finally, the conditions in Theorem 5.1 are derived by reversing the substitutions in (82) and (83). This completes the proof of the theorem. □

Appendix B

$$\left[\begin{array}{c|c|c|c} A & B_\Delta & B_1 & B_2 \\ \hline C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ \hline C_1 & D_{1\Delta} & D_{11} & D_{12} \\ \hline C_2 & D_{2\Delta} & D_{21} & 0 \end{array}\right] :=$$

$$\left[\begin{array}{ccccc|ccc|cc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
48.9844 & 0 & -48.9844 & 0 & 0 & 0 & -.35634 & -0.15548 & 0 & 0 & 0 \\
0 & 0 & 0 & .184940 & 0 & .0750596 & 0 & 0 & 0 & 0 & 50.0 \\
0 & 0 & 0 & -50.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -.50 & 0 & 0 & 0 & 0 & 0 & 0 & .50 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
436.33231 & 0 & -.043633 & 0 & 0 & 0 & 0 & .043633 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & .0036988 & 0 & .001501 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

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References

1. Anderson, B. and Vongpanitlerd, S. (1973), *Network analysis and synthesis: a modern systems theory*, Englewood Cliffs, NJ: Prentice-Hall.
2. Apkarian, P. and Gahinet, P. (1995), A Convex Characterization of Gain-Scheduled H_∞ Controllers, *IEEE Trans. Aut. Control* 40: 853–864. See also pp. 1681.
3. Apkarian, P., Gahinet, P. and Becker, G. (1995), Self- Scheduled H_∞ Control of Linear Parameter-Varying Systems: A Design Example, *Automatica* 31: 1251–1261.
4. Apkarian, P. and Tuan, H.D. (1998), Robust Control via Concave Minimization – Local and Global Algorithms, in *Proc. IEEE Conf. on Decision and Control*, Tampa, Florida.
5. Bennett, K.P. and Mangasarian, O.L. (1993), Bilinear Separation of Two Sets in n-Space, *Computational Optimization and Applications* 2: 207–227.
6. Boyd, S., ElGhaoui, E., Feron, E. and Balakrishnan, V. (1994), Linear Matrix Inequalities in Systems and Control Theory, vol. 15 of SIAM Studies in Applied Mathematics, SIAM, Philadelphia.
7. Coppel, W.A. (1974), Matrix quadratic equations, *Bull. Austral. Math. Soc.* 10: 377–401.
8. Doyle, J., Packard, A. and Zhou, K. (1991), Review of LFT's LMI's and μ, in Proc. IEEE Conf. on Decision and Control 2: 1227–1232.
9. Fan, M.K.H., Tits, A.L. and Doyle, J.C. (1991), Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics, *IEEE Trans. Aut. Control* AC-36: 25–38.
10. Gahinet, P. and Apkarian, P. (1994), A Linear Matrix Inequality Approach to H_∞ Control, *Int. J. Robust and Nonlinear Control* 4: pp. 421–448.
11. Gahinet, P., Nemirovski, A., Laub, A.J. and Chilali, M. (1995), LMI Control Toolbox, The MathWorks Inc.
12. Geromel, J.C., deSouza, C.C. and Skelton, R.E. (1998), Static Output Feedback Controllers: Stability and Convexity, *IEEE Trans. Aut. Control* 43: 120–125.
13. Ghaoui, L.E., Oustry, F. and Aitrami, M. (1997), A Cone Complementary Linearization Algorithm for Static Output-Feedback and Related Problems, *IEEE Trans. Aut. Control* 42: 1171–1176.
14. Goh, K.C., Safonov, G. and Papavassilopoulos, G.P. (1994), A Global Optimization Approach to the BMI Problem, in Proc. IEEE Conf. on Decision and Control 45–49.
15. Grigoriadis, K.M. and Skelton, R.E. (1996), Low-order Control Design for LMI Problems Using Alternating Projection Methods, *Automatica* 32: 1117–1125.
16. Helmersson, A. (1995), Methods for Robust Gain-Scheduling, Ph.D. Thesis, Linkoping University, Sweden.
17. Horst, R. and Pardalos, P. eds. (1995), Handbook of Global Optimization, Kluwer Academic Publishers.
18. Horst, R. and Tuy, H. (1996), Global Optimization: deterministic approaches, Springer (3rd edition).
19. Iwasaki, T. (1997), The dual iteration for fixed-order control, in Proc. American Control Conf., 1997, pp. 62–66.
20. Iwasaki, T. and Hara, S. (1996), Well-Posedness Theorem: A classification of LMI/BMI-reducible control problems, in Proc. Int. Symp. Intelligent Robotic Syst., Bangalore, pp. 145–157.

21. Iwasaki, T. and Skelton, R.E. (1994), All Controllers for the General H_∞ Control Problem: LMI Existence Conditions and State Space Formulas, *Automatica* 30: 1307–1317.
22. Kajiwara, H., Apkarian, P. and Gahinet, P. (1999), LPV techniques for control of an inverted pendulum, Control System Magazie, 19: 44–54.
23. Kokotovic, P.V. (1975), A Riccati equation for block-diagonalisation of ill-conditioned systems, IEEE Trans. Aut. Control, 20, 812–814.
24. Konno, H., Thach, P. and Tuy, H. (1997), Optimization on Low Rank Nonconvex Structures, Kluwer Academic Publishers.
25. Laub, A.J. (1979), A Schur method for solving algebraic Riccati equations, *IEEE Trans. Aut. Control* AC-24: 913–921.
26. Lyapunov, A.M. (1947), Problème général de la stabilité du mouvement, vol. 17 of Annals of Mathematics Studies, Princeton University Press, Princeton.
27. Mangasarian, O.L. and Pang, J.S. (1995), The Extended Linear Complementary Problem, *SIAM J. on Matrix Analysis and Applications* 16: 359–368.
28. Nesterov, Y.E. and Nemirovski, A.S. (1994), Interior Point Polynomial Methods in Convex Programming: Theory and Applications, SIAM Studies in applied Mathematics 13, SIAM, Philadelphia.
29. Packard, A. (1994), Gain Scheduling via Linear Fractional Transformations, *Syst Control Letters*, 22: 79–92.
30. Packard, A. and Becker, G. (1992), Quadratic Stabilization of Parametrically-Dependent Linear Systems using Parametrically-Dependent Linear, Dynamic Feedback, Advances in Robust and Nonlinear Control Systems, DSC-Vol. 43, pp. 29–36.
31. Packard, A., Zhou, K., Pandey, P. and Becker, G. (1991), A collection of robust control problems leading to LMI's in *Proc. IEEE Conf. on Decision and Control* 2: 1245–1250.
32. Scherer, G., Ganinet, P. and Chilali, M. (1997), Multi- Objective Output-Feedback Control via LMI Optimization, *IEEE Trans. Aut. Control* 42: 896–911.
33. Scorletti G. and Ghaoui, L.E. (1995), Improved Linear Matrix Inequality Conditions for Gain-Scheduling, in *Proc. IEEE Conf. on Decision and Control*, New Orleans, LA, 3626–3631.
34. Trentelman, H.L. and Willems, J.C. (1991), The dissipation inequality and the algebraic riccati equation, in The Riccati Equation, S. Bittani, A.J. Laub, and J.C. Willems, eds., Communications and Control Engineering Series, Springer-Verlag, ch. 8, 197–242.
35. Tuan, H., Apkarian, P., Hosoe, S. and Tuy, H., D.C. Optimization Approach to Robust Control: Feasibility Problems. To appear in Int. J. Control.
36. Tuan, H.D., Apkarian, P., Nakashima, Y., A new Lagrangian dual global optimization algorithm for solving bilinear matrix inequalities, in Proc. of 1999 American Control Conference, San Diego, CA, 1851–1855; Also to appear in Inf. J. of Nonlinear Robust Control.
37. Tuy, H. (1964), Concave Programming under linear constraints, Soviet Mathematics, 1437–1440.
38. Tuy, H. (1998), Convex analysis and global optimization, Kluwer Academic Publishers.
39. Yamada, Y. and Hara, S. (1996), The Matrix Eigenvalue Problem – Global Optimization for the Spectral Radius of a Matrix Product under Convex Constraints, in Proc. IEEE Conf. on Decision and Control, Kobe, 1326–1330.